## Part B Classical Mechanics: Problem Sheet 1 (of 4)

1. Consider a closed system consisting of $N$ point particles with masses $m_I$, position vectors $\mathbf{r}_I$ in an inertial frame $\mathcal{S}$, such that particle $J$ exerts a force $\mathbf{F}_{IJ}$ on particle $I$ for $I \neq J$.

   (a) Explain why Newton's third law and Galilean invariance means there exists an inertial frame $\mathcal{S}_0$ where the centre of mass

   $$\mathbf{R} \;=\; \frac{\sum_{I=1}^{N} m_I \mathbf{r}_I}{\sum_{I=1}^{N} m_I}$$

   is at rest at the origin.

   (b) Suppose now that there exist functions $V_{IJ} = V_{JI} = V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$, depending only on the distances $|\mathbf{r}_I - \mathbf{r}_J|$ between pairs of particles, such that $\mathbf{F}_{IJ} = -\partial_{\mathbf{r}_I} V_{IJ}$. Show that the total angular momentum and total energy

   $$E \;=\; \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 + \sum_{I<J} V_{IJ}(|\mathbf{r}_I - \mathbf{r}_J|)$$

   are conserved.

2. Consider the one-dimensional harmonic oscillator with action $S[q(t)] = \int_0^{2\pi} L(q,\dot{q})\,\mathrm{d}t$, Lagrangian

   $$L \;=\; L(q,\dot{q}) \;=\; \frac{1}{2}m\dot{q}^2 - \frac{1}{2}mq^2 \; ,$$

   and with boundary conditions $q(0) = q(2\pi) = 1$. Determine the Lagrange equation of motion, and show that a solution is $q(t) = \cos t$. Is this solution unique? By considering the two paths $q_1(t) = 1$, $q_2(t) = \cos 2t$ show that the critical function $q(t)$ is neither a maximum nor a minimum of the action $S$.

3. Consider the change of generalized coordinates $\mathbf{q} = \mathbf{q}(\tilde{\mathbf{q}}, t)$. Show that

   $$\dot{q}_a \;=\; \sum_{b=1}^{n} \frac{\partial q_a}{\partial \tilde{q}_b}\dot{\tilde{q}}_b + \frac{\partial q_a}{\partial t}.$$

   Defining

   $$\tilde{L}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t) \;\equiv\; L(\mathbf{q}(\tilde{\mathbf{q}}, t), \dot{\mathbf{q}}(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, t), t) \; ,$$

   show that the Lagrange equations in the two coordinate systems are related via

   $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial \tilde{L}}{\partial \tilde{q}_a} \;=\; \sum_{b=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_b}\right) - \frac{\partial L}{\partial q_b}\right]\frac{\partial q_b}{\partial \tilde{q}_a} \; .$$

   Hence conclude that the Lagrange equations take the same form in all coordinate systems.

4. Consider the purely kinetic Lagrangian

   $$L \;=\; T \;=\; \frac{1}{2}\sum_{a,b=1}^{n} g_{ab}(\mathbf{q})\dot{q}_a\dot{q}_b \; ,$$

where we assume that the symmetric matrix $g_{ab} = g_{ba}$ depends on the generalized coordinates $\mathbf{q}$ and is invertible at each point in configuration space. Show that Lagrange's equations take the form

$$\ddot{q}_a + \sum_{b,c=1}^{n} \Gamma^a_{bc}\dot{q}_b\dot{q}_c \;=\; 0 \;, \qquad a \;=\; 1,\ldots,n \;,$$

where

$$\Gamma^a_{bc} \;\equiv\; \frac{1}{2}\sum_{d=1}^{n}(g^{-1})^{ad}\left(\frac{\partial g_{bd}}{\partial q_c} + \frac{\partial g_{cd}}{\partial q_b} - \frac{\partial g_{bc}}{\partial q_d}\right) \;.$$

The matrix of functions $g_{ab}(\mathbf{q})$ defines a *metric* on configuration space, and the Lagrange equation is known as the *geodesic equation*.

5. Consider a system of two light rods of equal length, smoothly jointed together, with the other two ends of the rods fixed at two points in a horizontal plane. A mass $m$ is attached to the point where the rods are jointed. Introduce an appropriate generalized coordinate for the system, and determine the Lagrangian.

6. A *double pendulum* consists of a simple pendulum of mass $m_1$ and length $l_1$ pivoted at the origin, together with another simple pendulum of mass $m_2$ and length $l_2$, pivoted at the mass $m_1$. The whole system moves freely in a vertical plane under gravity. If $\theta_1$ and $\theta_2$ denote the angles each pendulum makes with the vertical, show that the Lagrangian is

$$\begin{aligned} L \;=\; & \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right] \\ & + m_1 g l_1\cos\theta_1 + m_2 g(l_2\cos\theta_2 + l_1\cos\theta_1) \;. \end{aligned}$$

7. (*Optional: this question is included for interest.*) Consider the central inverse square law force Lagrangian

$$L \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + \frac{\kappa}{r} \;,$$

where $r = |\mathbf{r}|$ and $\kappa$ is a constant.

   (a) Show by direct computation that the vector

   $$\mathbf{A} \;=\; \mathbf{p}\wedge\mathbf{L} - m\kappa\frac{\mathbf{r}}{r}$$

   is conserved, where $\mathbf{p} = m\dot{\mathbf{r}}$ and $\mathbf{L} = \mathbf{r}\wedge\mathbf{p}$ are momentum and angular momentum about the origin, respectively. [*Hint*: You may find it helpful to write $\dot{\mathbf{p}} = h(r)\frac{\mathbf{r}}{r}$ and show $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}\wedge\mathbf{L}) \;=\; -mh(r)r^2\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathbf{r}}{r})$.]

   (b) Show that $\mathbf{A}\cdot\mathbf{L} = 0$ and $|\mathbf{A}|^2 = 2mE|\mathbf{L}|^2 + m^2\kappa^2$, where $E$ is the conserved energy. Explain why $\mathbf{A}$ is a constant vector in the plane of the orbit.

   (c) By taking the dot product $\mathbf{A}\cdot\mathbf{r}$ derive the orbit equation

   $$\frac{1}{r} \;=\; \frac{m\kappa}{|\mathbf{L}|^2}\left(1 + \frac{|\mathbf{A}|}{m\kappa}\cos\theta\right) \;,$$

   where $\theta$ denotes the angle between $\mathbf{r}$ and $\mathbf{A}$. Notice we have found the orbit without solving any differential equation! The eccentricity is $|\mathbf{A}|/m\kappa$.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.